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THE TWO-FLUX MODEL FOR RADIATIVE TRANSFER WITH STRONGLY ANISOTROPIC SCATTERING

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1. INTRODUCTION

IN A RECENT examination of the two-flux model for radiative transfer in scattering systems, Brewster and Tien [1] concluded that when the scattering is acutely anisotropic the two-flux model is inaccurate, and furthermore that in order to obtain more accurate results it is necessary to resort to higher order flux models.

It is the purpose of this note to point out that the predictions of the two-flux model for the case of strongly anisotropic scattering may be considerably improved by making use of the delta-Eddington approximation [2], whereby the highly peaked scattering phase function is approximated as the sum of a forward-directed delta function and a linearly anisotropic phase function.

2. ANALYSIS

2.1. *The delta function approximation*

The equation of transfer for azimuthally symmetric radiation in an absorbing-scattering planar medium is [3]

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} = -I(\tau,\mu)+\frac{1}{2}\omega\int_{-1}^{+1} p(\mu,\mu')I(\tau,\mu')\,d\mu', \quad (1)$$

where $I(\tau,\mu)$ is the total intensity at optical depth τ in the direction μ and ω is the single-scatter albedo. The scattering phase function may be expanded in a series of Legendre polynomials as

$$p(\mu,\mu') = \sum_{n=0}^{\infty} a_n P_n(\mu)P_n(\mu'), \quad (2)$$

where P_n is the Legendre polynomial of degree n. The expansion coefficients a_n are calculated from Mie theory [4]. The phase function is normalized so that a_0 is unity.

The general form of the delta function method approximates the phase function as [5]

$$p(\mu,\mu') \approx 2\alpha\delta(\mu-\mu')+(1-\alpha)\sum_{n=0}^{N} a_n^* P_n(\mu)P_n(\mu'), \quad (3)$$

where

$$\alpha = a_{N+1}/(2N+3), \quad (4)$$

$$a_n^* = \frac{a_n-(2n+1)\alpha}{1-\alpha} \quad \text{for} \quad n = 0,\ldots,N. \quad (5)$$

For convenience, equation (3) is referred to in this paper as the $\delta-N$ approximation.

Substituting the approximate phase function of equation (3) into the radiation transfer equation yields

$$\mu\frac{\partial I(\tau^*,\mu)}{\partial\tau^*} = -I(\tau^*,\mu)+\frac{1}{2}\omega^*\int_{-1}^{+1} p^*(\mu,\mu')I(\tau^*,\mu')\,d\mu', \quad (6)$$

where

$$\tau^* = (1-\alpha\omega)\tau, \quad (7)$$

$$\omega^* = \frac{(1-\alpha)\omega}{1-\alpha\omega}, \quad (8)$$

$$p^*(\mu,\mu') = \sum_{n=0}^{N} a_n^* P_n(\mu)P_n(\mu'). \quad (9)$$

Thus, the delta function approximation leaves the form of the radiation transfer equation unchanged and scales the parameters τ, ω and $p(\mu,\mu')$ according to equations (7)–(9). Note that $a_0^* = 1$ so that p^* is a normalized phase function.

The simplest delta function approximation, known as the transport approximation [6], is obtained by taking $N = 0$ in equations (3) and (9). In this approximation the phase function is reduced to the sum of a forward-directed delta function and an isotropic function. The delta-Eddington approximation is obtained by taking $N = 1$. As noted above, in this approximation the phase function is replaced by the sum of a forward-directed delta function and a linearly anisotropic function $1+a_1^*\mu\mu'$. With both these approximations, the detailed structure (if any) in the exact phase function is lost. However, when only angle-integrated quantities are of interest, such as the net radiative flux, this loss is not significant because the calculation of angle-integrated quantities does not require precise detail in the scattering phase function. Of course, more of the structure of the phase function is retained in the higher orders of the approximation. The higher order $\delta-N$ approximations provide a convenient and accurate method for removing the forward diffraction (and refraction/reflection) peak from large-particle-scattering phase functions before applying the standard methods to solve the transfer equation.

2.2. *The two-flux model*

Two-flux models may be formulated by various methods. The traditional two-flux model is obtained by integrating the radiation transfer equation over forward and backward hemispheres in each of which the intensity is assumed to be isotropic. In the present work, an alternative two-flux model, based on the two-point discrete ordinate method [3], is obtained by evaluating the radiation transfer equation (6) at ordinates $\pm\bar{\mu}$, giving

$$\pm\bar{\mu}\frac{dI^\pm}{d\tau^*} = -I^\pm+\frac{1}{2}\omega^*[p^*(\pm\bar{\mu},\bar{\mu})I^+ + p^*(\pm\bar{\mu},-\bar{\mu})I^-]. \quad (10)$$

The choice $\bar{\mu} = 1/\sqrt{3}$ yields the so-called S_2 two-flux model, while the choice $\bar{\mu} = 1/2$ yields the DS_1 model which is substantially the same as the traditional two-flux model. The phase function $p^*(\mu,\mu')$ is unity for the $\delta-0$ approximation and $1+a_1^*\mu\mu'$ for the $\delta-1$ (delta-Eddington) approximation.

When a parallel beam of radiation is incident on the planar medium, as in the illustrative calculations presented below, it is convenient to recast the radiation transfer equation as an equation for the diffuse radiation intensity with the unscattered beam as a source term [3]. For the case of normal incidence, this re-formulation gives rise to an additional source term $(1/2)\omega^*p^*(\pm\bar{\mu},1)\,e^{-\tau^*}I_0$ on the RHS of equation (10),

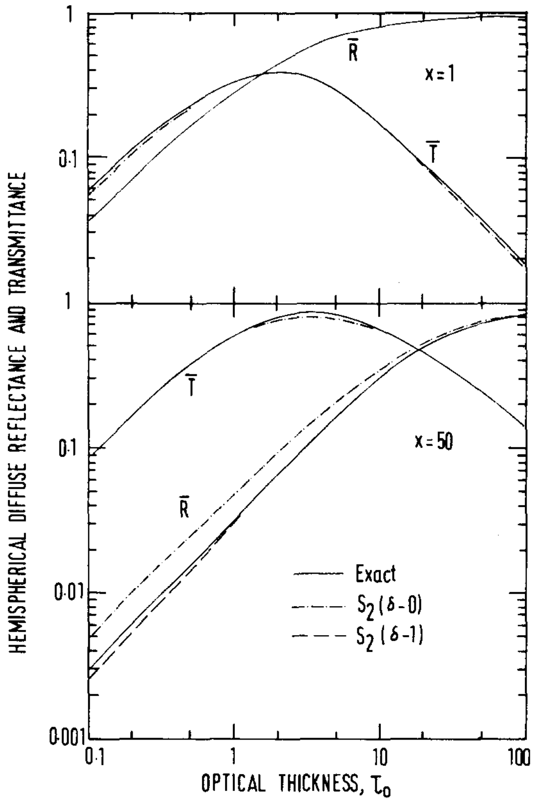

FIG. 1. Diffuse reflectance and transmittance as a function of optical thickness for particle size parameters 1 and 50.

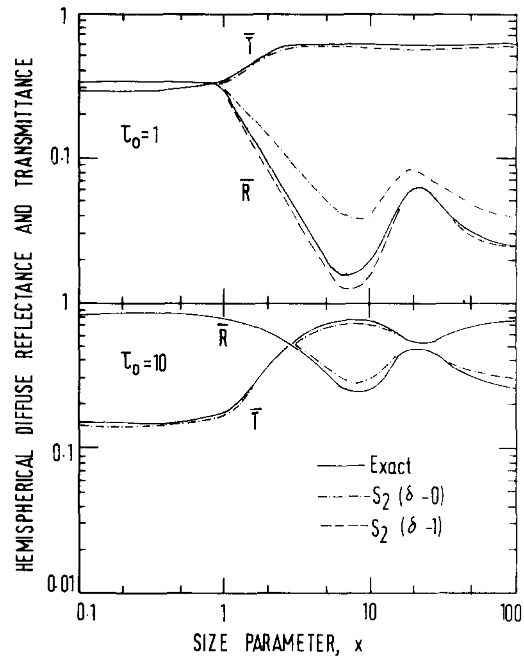

FIG. 2. Diffuse reflectance and transmittance as a function of particle size parameter for optical thicknesses 1 and 10.

where I_0 is the intensity of the incident radiation (at $\tau = 0$). The resulting pair of simultaneous equations (10) for the diffuse radiation intensities $I^{\pm}$ can be solved analytically. The diffuse reflectance and transmittance of the planar medium are given by

$$\bar{R} = \frac{\bar{\mu} I^{-}(0)}{I_0}, \tag{11}$$

$$\bar{T} = 1 - \bar{R} - e^{-\tau_0}, \tag{12}$$

where τ_0 is the optical thickness of the medium.

3. RESULTS AND DISCUSSION

Figures 1 and 2 show the normal-hemispherical diffuse reflectance and transmittance of a planar medium for the range of conditions examined by Brewster and Tien [1], namely, optical thicknesses τ_0 in the range 0.1–100, scattering albedo ω equal to unity (conservative medium), and phase function computed from the Mie theory [4, 7] for spherical particles with a refractive index n of 1.21 and particle size parameters x in the range 0.1–100. The 'exact' results were obtained by the method of discrete ordinates [3, 8] using a DS_{10} quadrature and where necessary (for $x \gtrsim 10$) the forward diffraction peak was removed from the phase function using the $\delta - 10$ approximation. The results were checked for accuracy using a higher order quadrature and a higher order delta function approximation. For particle size parameters greater than about 20, the diffuse reflectance and transmittance exhibit small oscillations with increasing particle size parameter. In this region the results have been smoothed to simplify the presentation.

Figure 1 shows $\bar{R}$ and $\bar{T}$ as functions of the optical thickness for fixed particle size parameters of 1 and 50. For $x = 1$, the phase function is not strongly anisotropic ($a_1 = 0.530$, $a_2 = 0.558$) and the $S_2(\delta-0)$ results are in good agreement with the exact results. However, for $x = 50$, the phase function is strongly anisotropic ($a_1 = 2.69$, $a_2 = 4.14$) and the $S_2(\delta-0)$ results for $\bar{R}$ are significantly in error, particularly at small values of τ_0. Figure 2 shows $\bar{R}$ and $\bar{T}$ as functions of the particle size parameter for fixed optical thicknesses of 1 and 10. It can be seen that for $x \gtrsim 1$, the $S_2(\delta-0)$ results are in error, particularly at $\tau_0 = 1$. The broad minimum in the value of $\bar{R}$ at $x \approx 8$ correlates with a minimum in the back-scatter fraction as a function of particle size parameter while the maximum at $x \approx 20$ correlates with a maximum in the back-scatter fraction [9]. The present results for the $S_2(\delta-0)$ model are generally similar to those reported by Brewster and Tien [1] for the traditional two-flux model and clearly demonstrate the deficiencies of the $S_2(\delta-0)$ model for strongly anisotropic scattering.

Also shown in Figs. 1 and 2 are the results for the $S_2(\delta-1)$ two-flux model. The improvement resulting from the use of the delta-Eddington approximation is obvious: in some regions the results of the $S_2(\delta-1)$ model and the exact results are indistinguishable.

4. SUMMARY AND CONCLUSIONS

Radiative transfer in an absorbing-scattering planar medium has been predicted using a two-flux model and compared with exact results. The traditional two-flux model is significantly in error when the scattering is strongly anisotropic. In this case, a considerable improvement in the accuracy of the two-flux model may be achieved by making use of the delta-Eddington approximation for the phase function.

REFERENCES

1. M. Q. Brewster and C. L. Tien, Examination of the two-flux model for radiative transfer in particulate systems, *Int. J. Heat Mass Transfer* **25**, 1905–1907 (1982).
2. J. H. Joseph, W. J. Wiscombe and J. A. Weinman, The delta-Eddington approximation for radiative flux transfer, *J. Atmos. Sci.* **33**, 2452–2459 (1976).
3. S. Chandrasekhar, *Radiative Transfer*. Dover, New York (1960).

4. H. C. Van de Hulst, *Light Scattering by Small Particles*. Wiley, New York (1957).
5. W. J. Wiscombe, The delta-M method. Rapid yet accurate radiative flux calculations for strongly asymmetric phase functions, *J. Atmos. Sci.* **34,** 1408–1422 (1977).
6. B. Davison, *Neutron Transport Theory*. Oxford University Press, London (1958).
7. G. W. Kattawar and G. N. Plass, Electromagnetic scattering from absorbing spheres, *Appl. Optics* **6,** 1377–1382 (1967).
8. T. J. Love and R. J. Grosh, Radiative heat transfer in absorbing, emitting and scattering media, *J. Heat Transfer* **C87,** 161–166 (1965).
9. W. J. Wiscombe and G. W. Grams, The backscatter fraction in two-stream approximations, *J. Atmos. Sci.* **33,** 2440–2451 (1976).

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MULTI-PRANDTL NUMBER CORRELATION EQUATIONS FOR NATURAL CONVECTION IN LAYERS AND ENCLOSURES

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INTRODUCTION

An earlier note [1] reported an expression which closely fits the experimental *Nu–Ra* data relevant to natural convection in horizontal heated-from-below layers of a fluid (water) having a Prandtl number of about 6. The present note has two purposes: (1) to point out that a slightly altered form of that expression closely fits all the reliable available experimental data relevant to horizontal layers, regardless of the Prandtl numbers; and (2) to point out that a slightly altered form of this new expression fits the available experimental data relevant to horizontal *enclosures* as well as layers. (For the purposes of this note a horizontal enclosure is one bounded only by horizontal and vertical surfaces, and a horizontal layer is a horizontal enclosure whose horizontal dimensions have been made so large with respect to the vertical ones that they have ceased to effect the Nusselt number. See the earlier note [1] for definitions of *Nu* and *Ra*.)

HORIZONTAL LAYERS

The earlier note [1] gave *Nu–Ra* expressions for both air ($Pr \approx 0.7$) and water ($Pr \approx 6$) and demonstrated a definite Prandtl number dependence inside the range $0.7 \lesssim Pr \lesssim 6$. That the Prandtl number dependence persists outside that range is demonstrated in Fig. 1, which shows the data of Schmidt and Silveston [2] at $Pr = 35$, 100, and 3000, Rossby [3] at $Pr = 200$ and 0.025, and Globe and Dropkin [4] at various *Pr*, together with plots of the earlier note's expressions for air and water. The consistency of the Schmidt and Silveston data with the Rossby data (or vice versa) is both striking and reassuring. But the Globe and Dropkin data is generally inconsistent with the other data. A study of the Globe and Dropkin experiment reveals sources for experimental error large enough to explain the inconsistency;* hence their data will be ignored in what follows.

The proposed altered form of the previous note's expression is

$$Nu = 1+[1-1708/Ra]^{\bullet}[k_1+2(Ra^{1/3}/k_2)(1-\ln(Ra^{1/3}/k_2))] + [(Ra/5830)^{1/3}-1]^{\bullet}, \quad (1)$$

* For example the hot plate was not guarded on either the bottom or the sides, and no special precautions were taken to ensure the isothermality of each plate.

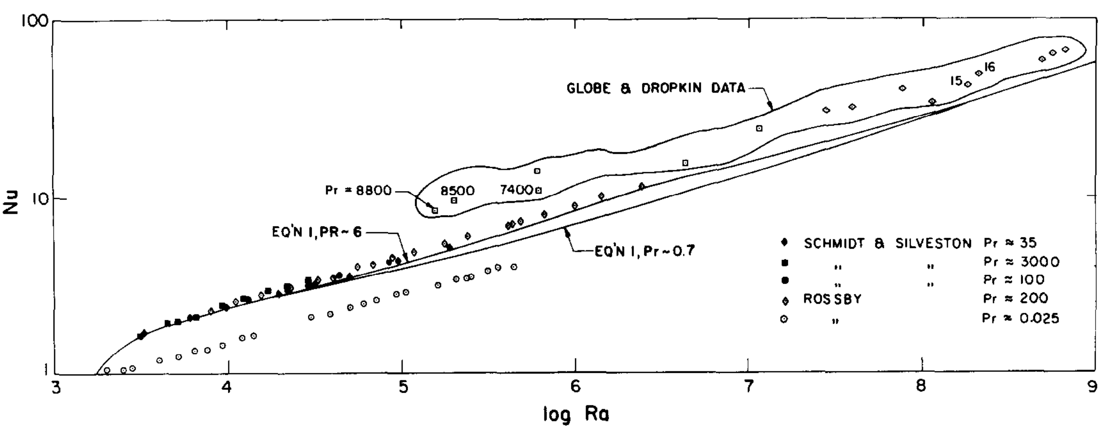

Fig. 1. Data of various workers for Prandtl numbers outside the range $0.7 < Pr < 6$. Also shown are the fits for the data at $Pr = 0.7$ and 6 obtained in the earlier note [1].